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LETTER TO THE EDITOR

Backbones of traffic jams

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Abstract. We study the jam phase of the deterministic traffic model in two dimensions. Within the jam phase, there is a phase transition, from a self-organized jam (formed by initial synchronization followed by jamming), to a random-jam structure. The *backbone* of the jam is defined and used to analyse self-organization in the jam. The fractal dimension and interparticle correlations on the backbone indicate a continous phase transition at density p_c with critical exponent μ, which are characterized through simulations.

Models of traffic flow that have recently been studied extensively [1–3] have revealed a wealth of interesting behaviour such as phase transitions and self-organization [4], kinematic waves [5] and long-range spatio-temporal correlations, evidenced, for example as $1/f$ noise [6, 7]. One of the simplest of such models is the deterministic cellular automaton (CA), introduced by Biham *et al* [2] (BML). This model for traffic flow in two dimensions appears to show a kinetic first-order transition from a low-density freely moving phase to a high-density jammed or blocked phase. Mean-field treatments of this model [8] also predict a transition from a moving to a jammed phase. The robustness of this jamming transition is unclear, and indications are that similar transitions in related systems [9, 10] may not survive in the thermodynamic limit [11].

The jammed phase, on the other hand, has been shown to have structure: the very high-density jam is a *random* structure with exponentially suppressed correlations, while at lower densities, the jam is *self-organized*, with long-ranged spatial correlations. By calculating the diagonal correlation functions and fitting a power-law behaviour to them, Tadaki and Kikuchi [12] reported the transition between a self-organized and a random jam at $p = 0.52$.

In this letter we study the transition between the self-organized and the random phases of the traffic jam in the BML model to elucidate its existence and nature. We identify an organizing subset of the jammed phase which reveals the change from a self-organized jam to a random jam and study spatio-temporal correlations, as well as the kinetics leading to the two different phases.

In the jam phase, there is a single cluster that spans the entire system. We examine the backbone of this traffic jam, which is the *essential* structure preventing motion of cars. Our results on the behaviour of the fractal characteristics of the backbone, and the interparticle correlations on the backbone show a *continuous* transition between the self-organized and random jam, and also clearly rule out any transition at $p = 0.52$ as reported in [12].

The deterministic traffic model introduced by Biham *et al* [2] is, like other CA models of traffic flow, essentially a lattice gas system with two species of particles. It is defined as follows. On a $L \times L$ square lattice in two-dimensions, there are $n_\uparrow$ cars which move

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only upwards (S–N) and $n_{\rightarrow}$ cars which move only rightward (W–E), distributed at random sites. The dynamical rules are simple:

(i) a car can only move to a neighbouring empty site;

(ii) $\uparrow$ and $\rightarrow$ cars move at alternate time steps;

(iii) all cars of a given type attempt to move simultaneously (parallel updating).

The density of cars is $p = p_{\uparrow} + p_{\rightarrow} = (n_{\uparrow} + n_{\rightarrow})/L^2$, and the mean velocity of the system is the fraction of cars that move in two time steps, $\langle v(t) \rangle = n_{\mathrm{mov}}/n_{\mathrm{tot}}$, where $n_{\mathrm{tot}} = n_{\uparrow} + n_{\rightarrow}$. The isotropic case $n_{\uparrow} = n_{\rightarrow} = pL^2/2$ is considered.

At low density, all cars eventually move [2, 13] and $v \rightarrow 1$: this is the moving phase. In this phase there is self-organization along the NW–SE diagonals, each of which is occupied only by cars of a single type, to avoid collisions in the freely flowing phase [13]. In a typical moving phase, alternate diagonals are occupied by cars of opposite type, and the diagonals flow freely, having been synchronized to avoid collisions as explained above.

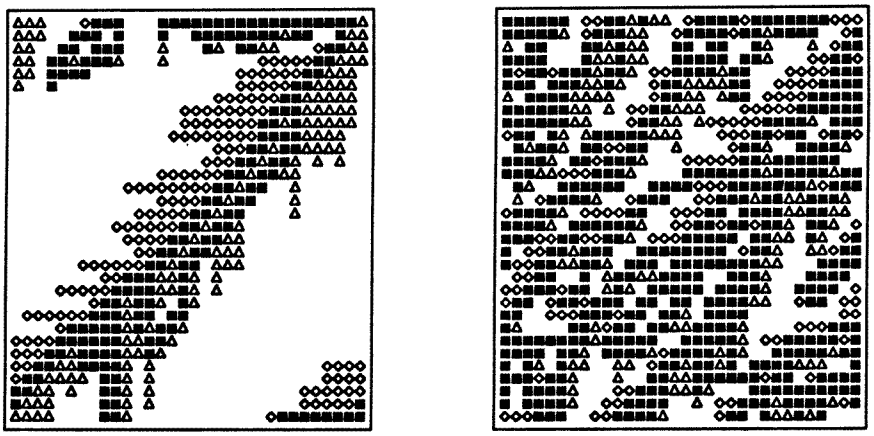

Figure 1. Typical jamming configuration in a 32×32 system in the (a) low ($p = 0.45$) and (b) high ($p = 0.80$) density phase. Backbone cars are shown as □s, dangling $\rightarrow$ cars as ◇s, and dangling $\uparrow$ cars as △s.

Upon increasing the density of cars p above the system size dependent p_{t} ($p_{\mathrm{t}} = 0.315$ for 512×512 lattices) there is a *jamming transition* when the system eventually reaches a configuration when no car can move: all cars are immobile in a single global cluster which spans the lattice. In this phase, $v \rightarrow 0$, and the jam extends across the SW–NE diagonals of the lattice. The abrupt nature of the transition from $v = 1$ to $v = 0$ indicates a dynamical first-order transition. The nature of the jammed phases shows a gradual change from a single strip, nearly diagonal band of cars near p_{t}, (see figure 1(a)) to a multistrip, branched structure for higher densities (See figure 1(b)). We examine this transition by considering the backbone of the jam, which is described below.

Local jamming occurs when a car is blocked by another from moving ahead. Small local jams form which grow in size, above a critical jam size, as the rate of car pile-up exceeds the rate at which cars leave the upper-right region of the jam. The typical jam consists of a backbone of cars which organize the jam structure, and much larger numbers of dangling cars accumulated around them. To find the backbone of the jam, we reverse the process by which dangling cars accumulate around the backbone, by iteratively removing

those cars on the periphery of the cluster whose removal does not trigger an instability in the jam [14].

We find the backbone to be fractal, both in the low-density self-organized regime when the backbone is nearly diagonal (see figure 1(*a*)) or in the high-density random regime, when it is extensively ramified (see figure 1(*b*)). The (capacity) fractal dimension of the backbone, ν, is defined through

$$M_{\mathrm{B}} = \ell^{\nu} \tag{1}$$

where M_{B} is the number of backbone cells at scale ℓ.

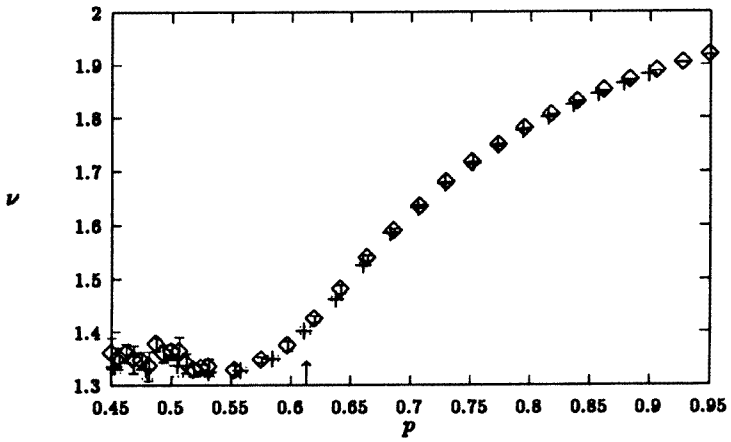

Figure 2. Plot of the fractal dimension ν as function of density above the threshold. A clear phase transition occurs at $p_{\mathrm{c}} = 0.59 \pm 0.02$. Our results are for lattices of size 256 ($\Diamond$) and 512 ($+$), and are averaged over several realizations.

Figure 2 shows the variation of the backbone fractal dimension with density. For p around p_{t}, the fractal dimension remains almost constant, but with increasing density, there is a clear change observed for p above a threshold p_{c}. This threshold is independent of system size, reminiscent of a second-order phase transition. The smooth nature of the transition indicates that self-organization will be noticeable at densities higher than p_{c}, as is indeed observed. Least-squares fits [15] to the form $\nu \sim (p - p_{\mathrm{c}})^{\mu}$ gives $\mu = 0.52 \pm 0.02$, while the threshold density is determined as $p_{\mathrm{c}} = 0.59 \pm 0.02$ [16]. Our simulations were carried out on lattices of size $L = 256$, 512 and 1024, and the critical point p_{c} was observed to be independent of system size. The power-law dependence on p suggests a continous phase transition. It should be noted that the same transition point can also be obtained by looking at the entire jam as in [17], where the transition density was calculated to be near the two-dimensional percolation threshold.

We now investigate why the jams in the low- and high-density phases are, respectively, self-organized and random. Define the pair correlation function as

$$C_{\alpha\beta}(\vec{r}) = \sum_{\vec{x}} \rho_{\alpha}(\vec{x})\rho_{\beta}(\vec{r} + \vec{x}) \qquad \alpha, \beta \equiv \uparrow, \rightarrow \tag{2}$$

where $\rho_{\alpha}(\vec{r})$, the density of cars of type α at site $\vec{r}$, is one if that site is occupied by a car of type α and zero otherwise. In the limit of large r, $\rho_{\alpha}(\vec{x} + \vec{r}) \approx \rho_{\alpha}(\vec{r})$, which

approaches the equilibrium car density $\rho_\alpha(\vec{r}) = p/2$ independent of site $\vec{r}$. Therefore, $C_{\alpha\beta}(r) \to n_\alpha \rho_\beta = n_\alpha p/2$ for large r.

Backbone correlation functions can be defined as in equation 2:

$$\mathcal{C}_{\alpha\beta}(\vec{r}) = \sum_{\vec{x}}{}' \rho_\alpha(\vec{x})\rho_\beta(\vec{r}+\vec{x}) \tag{3}$$

with the prime on the summation indicating that the sum is only over sites $\vec{x}$ on the backbone. In the jam, the number of cars of both species are equal, but on the backbone these can differ (although the difference is usually small, by isotropy). The limiting values for the correlation functions for the backbone are therefore

$$\mathcal{C}_{\uparrow\uparrow}(\vec{r}) = L^2 \mathcal{P}_\uparrow \mathcal{P}_\uparrow \tag{4}$$

$$\mathcal{C}_{\uparrow\rightarrow}(\vec{r}) = L^2 \mathcal{P}_\uparrow \mathcal{P}_\rightarrow \tag{5}$$

(where $\mathcal{N}_\alpha$ and $\mathcal{P}_\alpha$ refer to the number and density of α type cars on the *backbone* only) which are normalized as follows: $\mathcal{C}_{\uparrow\uparrow}(\vec{r}) \to \mathcal{C}_{\uparrow\uparrow}(\vec{r})/\mathcal{N}_\uparrow, \mathcal{C}_{\uparrow\rightarrow}(\vec{r}) \to \mathcal{C}_{\uparrow\rightarrow}(\vec{r})/\mathcal{N}_{\rm avg}$ where $\mathcal{N}_{\rm avg} = (\mathcal{N}_\uparrow + \mathcal{N}_\rightarrow)/2$. Neglecting the small difference in the number of $\uparrow$ and $\rightarrow$ cars on the backbone, $\mathcal{C}_{\uparrow\uparrow}(\vec{r}) \to \mathcal{P}_\uparrow$ and $\mathcal{C}_{\uparrow\rightarrow}(\vec{r}) \to \mathcal{P}_{\rm avg} = \mathcal{N}_{\rm avg}/L^2$ for large r.

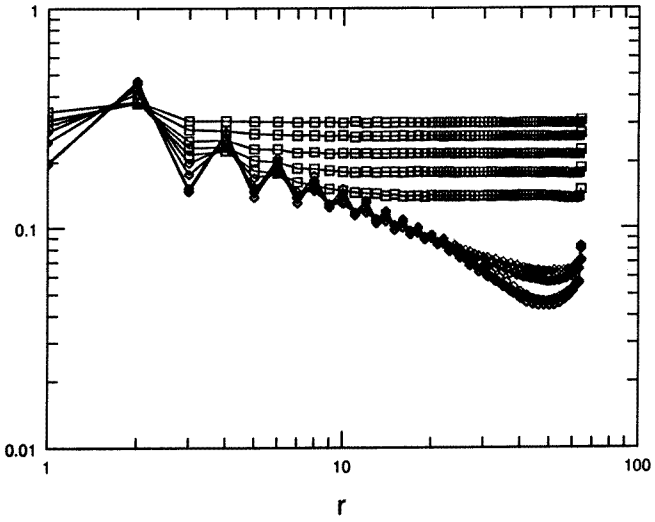

Figure 3. $\mathcal{C}_{\uparrow\uparrow}(r)$ for different densities. The lines, for $L = 64$, from bottom to top correspond to the densities $p = 0.85, 0.80, 0.75, \ldots, 0.45$. The lower densities from $p = 0.45$–0.60 comprise the self-organized phase, and car densities above $p = 0.65$ constitute the random phase. Self organized phase: $\diamondsuit$, and random phase: $\square$.

Shown in figure 3 are the averaged backbone self-correlation functions for up moving cars, $\mathcal{C}^{\rm av}_{\uparrow\uparrow}(r)$, where the averaging is performed over sites such that $x + y = r$. For densities below $p_{\rm c}$, $\mathcal{C}^{\rm av}_{\uparrow\uparrow}(r)$ is independent of density p, implying that the backbone has a spatially similar structure throughout the low-density phase. For $p > p_{\rm c}$, the limiting value of the correlation functions increases with density: the number of cars belonging to the backbone increases steadily above $p_{\rm c}$, in agreement with the results on the fractal dimension which show the mass of the backbone increasing steadily above $p_{\rm c}$. A plot of the limiting value $\mathcal{P}_\alpha$ of $\mathcal{C}^{\rm av}_{\uparrow\uparrow}(r)$ (see figure 4) shows an alternate means of detecting the phase transition type of behaviour.

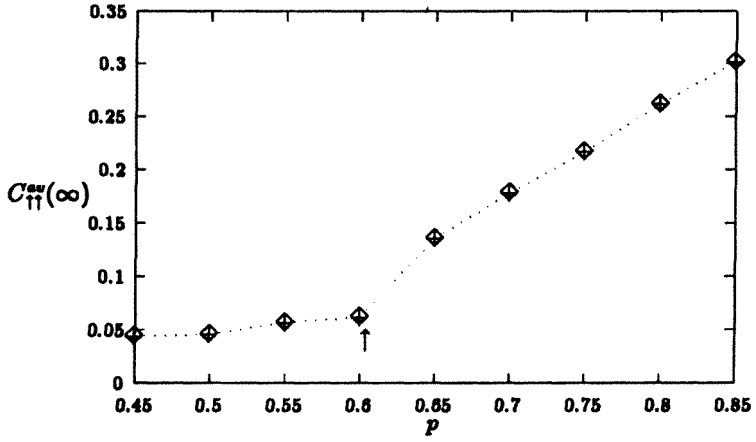

Figure 4. Limiting value of $\mathcal{C}^{\rm av}_{\uparrow\uparrow}(r)$ (taken at $r = L/2$) versus p for a 64×64 lattice. The break in behaviour above and below $p_{\rm c} = 0.59 \pm 0.02$ is similar to figure 2, and is another way of detecting the transition.

The lowest density configuration at which a jam may occur in the BML model consists of a single globally connected zigzag jam extending across the system ($p = 2L/L^2$). A zigzag jam is a diagonal band of up and right moving cars in which each up mover is blocked by a right mover above it, and a right mover blocked by an up mover to its right. It is clear (see figure 1(a)) that the backbone of the jam for lower densities consists mainly of these zigzag jams. For higher densities, branching and off-diagonal structures become dominant and zigzag jams no longer form the largest section of the jam. To see this distinction, one can study the self-correlation function $\mathcal{C}^{\rm av}_{\uparrow\uparrow}(r)$ shown in figure 3, in the high- and low-density regimes. The cross-correlation function $\mathcal{C}^{\rm av}_{\uparrow\rightarrow}(r)$ (not shown) has a similar structure, shifted right by one unit.

In the low p region where the system is largely dominated by zigzag jams, the (full) correlation functions $\mathcal{C}_{\alpha\beta}(\vec{r})$ have their largest values on a narrow band along the diagonal, because of the self-organized nature of the zigzag jam. It can be seen that the autocorrelation $\mathcal{C}_{\uparrow\uparrow}(\vec{r})$ get a large contribution along $\vec{r}_m = m(\hat{\imath}+\hat{\jmath})$, $m = 1, \ldots, N$, and the cross correlation $\mathcal{C}_{\uparrow\rightarrow}(\vec{r})$ get a large contribution along $\vec{r}^{\,'}_m = \hat{\jmath} + \vec{r}_m$. It follows that both the averaged correlations will display oscillatory structure. Maxima occur at $r = 2k, 2k = 0, \ldots, L$ for $\mathcal{C}^{\rm av}_{\uparrow\uparrow}(r)$, and at $r = 2k-1, 2k-1 = 1, \ldots, L-1$ for the cross correlations $\mathcal{C}^{\rm av}_{\uparrow\rightarrow}(r)$. Our results follow this behaviour, and fall on approximately the same curve. Finally, for higher densities, the jam structure forms randomly, without self-organization, and oscillations are suppressed (see figure 3). By including off-diagonal effects, our results on the correlation functions are a cleaner and more graphic way of identifying the self-organized to random transition, as opposed to the diagonal correlation function calculated in Tadaki and Kikuchi's paper [12].

Below $p_{\rm c}$, when the jam forms extremely slowly [2], for long times the cars are nearly synchronized in an almost freely moving phase, with most of the cars organized into freely flowing diagonals (FFD) along the SE–NW directions. This was first pointed out in the original paper by Biham *et al* [2], and explained theoretically by Shi [13]. When the jam occurs, it is triggered by isolated asynchronous cars, which block the motion of a FFD of opposite car type for sufficient time to desynchronize its motion with respect to

its immediate neighbour FFDs, leading to formation of long zigzag jams. This correlation present in the pre-equilibration phase thus eventually manifests itself in the final jam in the form of long, self-organized zigzag jams [18]. In the high density phase, cars jam locally before correlating their motion globally, leading to a jam with random structure.

The significance of the critical density $p_c \approx \frac{3}{5}$ is not clear, and seems to be unrelated to the maximum density for which a flowing phase is possible ($p_{max} = \frac{2}{3}$) [2, 13].

In summary, we see a clear indication of a transition between the self-organized jam and the random jam in the BML model, by identifying the *backbone* along which the jam organizes. The variation of the fractal dimension of the backbone gives evidence for a continous phase transition at a critical density $p_c = 0.59 \pm 0.02$ with a critical exponent $\mu = 0.52 \pm 0.02$. The same transition point and exponent can be obtained by looking at the fractal characteristics of the total jam, and not just the backbone [17]. However, by focusing on the backbone, a cleaner signature for the phase transition is obtained and the spatial correlations on the backbone clearly show a change from the self-organized to the random structure, at p_c. The backbone analysis also rules out a transition at $p = 0.52$, which was identified by Tadaki and Kikuchi in [12] as the transition density above which self-organization is suppressed. The change in behaviour above p_c is explained in terms of a synchronization versus jamming scenario, in which the correlations in the pre-equilbration phase manifest themselves as self-organization in the final jammed phase.

This continous transition is more robust than the first-order jamming transition itself, in the following sense. There exists a zero density jamming configuration with $\nu = 1$, while for very high densities, $\nu \to 2$. Since for $p \geqslant p_{min} = 2L/L^2$, the fractal dimension ν cannot change discontinously to 2, it is highly probable that the transition is robust, unlike the jamming transition where simulations [2], and theoretical treatments [11] indicate that the transition point decreases with increasing system size, eventually vanishing in the thermodynamic limit. It may be noted, however, that the improved mean-field treatment of the model given by Wang *et al* [8] predicts a system size independent transition at $p = 0.343$, close to the critical point $p = 0.315$ obtained by simulations on the largest systems (512×512).

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Note added in proof. The following papers:

1. Török J, Kertész J and Manna S S 1996 Phase transition in a two-dimensional green wave traffic model *Proc. Conf. on Traffic and Granular Flow (Jülich, 1995)* ed M Schreckenberg and D E Wolf (Singapore: World Scientific)

2. Török J and Kertész J 1996 The green wave model of two-dimensional traffic: Transitions in the flow properties and in the geometry of the traffic jam *Physica* A **231** 515–33

describe the jam phases in a related deterministic traffic model.

References

[1] Nagel K and Schreckenberg M 1992 *J. Physique* **2** 2221
Nagatani T 1993 *J. Phys. Soc. Japan* **62** 1085; *J. Phys. A: Math. Gen.* **26** L1015
Bussemaker H J 1996 *Phys. Rev.* E **53** 1644
Helbing D 1996 *Phys. Rev.* E **53** 2366

[2] Biham O, Middleton A and Levine D 1992 *Phys. Rev.* A **46** R6124

[3] Cuesta J A, Martínez F C, Molera J M and Sanchez A 1993 *Phys. Rev.* E **48** R4175
Martínez F C, Cuesta J C, Molera J M and Brito R 1995 *Phys. Rev.* E **51** R835
Molera J M, Martínez F C, Cuesta J A and Brito R 1995 *Phys. Rev.* E **51** 175

[4] Csańyi G and Kertész J 1995 *J. Phys. A: Math. Gen.* **28** L427
[5] Lighthill M J and Whitham G B 1955 *Proc. R. Soc.* A **229** 281, 317. See also Whitham G B 1974 *Linear and Nonlinear Waves* (New York: Wiley)
[6] Nagel K and Paczuski M 1995 *Phys. Rev.* E **51** 2909
[7] Choi M Y and Lee H Y 1995 *Phys. Rev.* E **52** 5979
[8] Nagatani T 1993 *J. Phys. Soc. Japan* **62** 2656
Ishibashi Y and Fukui M 1994 *J. Phys. Soc. Japan* **63** 2882
Wang B H, Woo Y F and Hui P M 1996 *J. Phys. A: Math. Gen.* **29** L31
[9] Schmittmann B, Hwang K and Zia R K P 1992 *Europhys. Lett.* **19** 19
Bassler K F, Schmittman B and Zia R K P 1993 *Europhys. Lett.* **24** 115
[10] Kertész J and Ramaswamy R 1994 *Europhys. Lett.* **28** 617
[11] Foster D P and Godrèche C 1994 *J. Stat. Phys.* **76** 1129
[12] Tadaki S and Kikuchi M 1994 *Phys. Rev.* E **50** 4564
[13] Shi Y 1995 Self-organisation, ergodicity breaking and synchronisation in two-dimensional traffic-flow model *Preprint* October 1995, LANL Bulletin Board, adap-org 9509003
Kaulke M and Trimper S 1995 *J. Phys. A: Math. Gen.* **28** 5445
[14] This is similar to the 'burning algorithm' used in percolation studies
[15] Press W H, Teukolsky S A, Vetterling W T and Flannery B P 1992 *Numerical Recipes in Fortran : The Art of Scientific Computing* 2nd edn (Cambridge: Cambridge University Press)
[16] To fit data to a power-law behaviour, an estimate of the transition point p_c is used in a nonlinear least-squares procedure
[17] Tadaki S and Kikuchi M 1995 *J. Phys. Soc. Japan* **64** 4504
[18] For the BML model, it is possible to define an order parameter $\phi(t)$ which measures the degree of segregation into these single species FFDs and our results indicate that the segregation is complete in the very low-density regime, while for $p \approx p_t$ there is a pre-equilibration phase in which the FFDs form, followed by jamming, when $\phi(t)$ drops to near zero.